Normalization conditions at $k \neq 0$:
(here we specify to $m=0$)

$$\Gamma_R^{(2)}(0;g) = 0 \qquad (1a)$$

$$\frac{\partial}{\partial k^2}\Gamma_R^{(2)}(k;g)\Big|_{k^2=\kappa^2} = 1 \qquad (1b)$$

$$\Gamma_R^{(4)}(k_i;g)\Big|_{SP} = g \qquad (1c)$$

where SP is symmetry point:

$$k_i \cdot k_j = \frac{\kappa^2}{4}(4\delta_{ij}-1)$$

$$\longrightarrow p^2 = (k_1+k_2)^2 = \underbrace{k_1^2+k_2^2}_{=2\cdot\frac{3}{4}\kappa^2} + \underbrace{2k_1k_2}_{=-\frac{1}{4}\kappa^2}$$

$$= \kappa^2$$

i.e. total incoming momentum fixed.

Now take

$$\Gamma_R^{(N)}(k_i;g(\kappa_1),\kappa_1) = Z_\phi^{N/2}\,\Gamma^{(N)}(k_i;\lambda,\Lambda) \qquad (2)$$

such that g in (1) is determined at momentum κ_1.

$$Z_\phi = Z_\phi(g(\kappa_1),\kappa_1,\Lambda)$$

$\Gamma_R^{(N)}$ is limit of right-hand-side for $\Lambda\to\infty$

$\longrightarrow$ Z_ϕ diverges logarithmically in $d=4$

If we renormalize instead at κ_2, we get

$$\Gamma_R^{(N)}(k_i, g(\kappa_1), \kappa_1) = \left[Z(\kappa_2, g_2, \kappa_1, g_1)\right]^{N/2} \Gamma_R^{(N)}(k_i; g(\kappa_2), \kappa_2) \quad (3)$$

where we use $g_i \equiv g(\kappa_i)$ and

$$Z(\kappa_2, g_2, \kappa_1, g_1) = Z_\phi(g_1, \kappa_1, \Lambda) / Z_\phi(g_2, \kappa_2, \Lambda)$$

is finite in the limit $\Lambda \to \infty$

Furthermore,

$$\frac{\partial}{\partial k^2}\left(\left[Z\right]^{N/2} \Gamma_R^{(N)}(k_i, g_2, \kappa_2)\right)\Big|_{k^2=\kappa_2^2}$$

$$= \left[Z\right]^{N/2} \underbrace{\left(\frac{\partial}{\partial k^2} \Gamma_R^{(N)}(k_i; g_2, \kappa_2)\Big|_{k^2=\kappa_2^2}\right)}_{\overset{(1b)}{=}\ 1}$$

set $N=2$

$= \quad Z$

Thus $\quad Z(\kappa_2, g_2, \kappa_1, g_1) = \dfrac{\partial}{\partial k^2} \Gamma_R^{(2)}(k; g_1, \kappa_1)\Big|_{k^2=\kappa_2^2}$

$$= Z(\kappa_2, g_1, \kappa_1) \quad (4)$$

(1c) at $k_i = Sp(\kappa_2)$ gives

$$g_2 = \left[Z(\kappa_2, \kappa_1, g_1)\right]^{-2} \Gamma_R^{(4)}(k_i; g_1, \kappa_1)\Big|_{k_i = Sp(\kappa_2)} = R(\kappa_2, \kappa_1, g_1) \quad (5a)$$

with R satisfying

$$R(\kappa, \kappa, g) = g \qquad (5b)$$

Definition:

The renormalization group is a group of transformations τ_i under which the normalization momentum is multiplied by real positive number t_i. $\longrightarrow$ under $\tau_1 * \tau_2$ we get the scaling by $t_1 t_2$.

$$\Gamma_R^{(N)}(k_i; g_1, \kappa_1) = Z^{N/2}\,\Gamma_R^{(N)}(k_i; R(\kappa_2, \kappa_1, g_1), \kappa_2)$$

"functional equation of the group"

Derivation of corresponding differential eq:

In the limit $\Lambda \to \infty$:

$$\left(\kappa \frac{\partial}{\partial \kappa}\right)_{\lambda,\Lambda}\left[Z_\phi^{-N/2}\,\Gamma_R^{(N)}(k_i; g, \kappa)\right] = 0$$

($\Gamma^{(N)}$ independent of κ)

Rewrite as follows

(6) $$\left[\kappa \frac{\partial}{\partial \kappa} + \bar{\beta}(g,\kappa)\frac{\partial}{\partial g} - \frac{1}{2} N \gamma_\phi(g,\kappa)\right]\Gamma_R^{(N)}(k_i; g, \kappa) = 0$$

where $$\bar{\beta}(g,\kappa) = \left(\kappa \frac{\partial}{\partial \kappa} g\right)_{\lambda,\Lambda}$$

and $$\gamma_\phi(g,\kappa) = \left(\kappa \frac{\partial \ln Z_\phi}{\partial \kappa}\right)_{\lambda,\Lambda}$$

$\bar{\beta}$ and γ_ϕ are finite functions in limit $\Lambda \to \infty$

as β can be obtained from eqs. (5a), (5b) and γ_ϕ from (3) and (4).

Now λ and g have dimensions Λ^{4-d}

$$\longrightarrow \lambda = u_0 \kappa^{\varepsilon}, \quad g = u\kappa^{\varepsilon}$$

Then eq. (6) becomes

$$\left[\kappa \frac{\partial}{\partial \kappa} + \beta(u)\frac{\partial}{\partial u} - \frac{1}{2} N \gamma_\phi(u)\right] \Gamma_R^{(N)}(k_i; u, \kappa) = 0 \qquad (7)$$

where $\kappa \frac{\partial}{\partial \kappa}$ is evaluated at constant u (previously constant g in (6))

and we have

$$\beta(u) = \left(\kappa \frac{\partial u}{\partial \kappa}\right)_\lambda$$

$$\gamma_\phi(u) = \kappa \left(\frac{\partial \ln Z_\phi}{\partial \kappa}\right)_\lambda$$

Note the following identity

$$\kappa \left(\frac{\partial u}{\partial \kappa}\right)_\lambda = - \frac{\kappa(\partial\lambda/\partial\kappa)_u}{(\partial\lambda/\partial u)_\kappa} \qquad (8)$$

From dimensional analysis, we have

$$\lambda = \kappa^{\varepsilon} u_0(u, \kappa/\Lambda)$$

as u_0 is dimensionless

u_0 is finite for $d \to 4$ ($\varepsilon \to 0$), Λ fixed

" " " $d < 4$, $\Lambda \to \infty$

β is finite for $\varepsilon \to 0$, $\Lambda \to \infty$

Calculate (8) by holding $\varepsilon > 0$ constant and sending $\Lambda \to \infty$ giving $u_0 = u_0(u)$

will have poles at $\varepsilon = 0$

We compute

$$\left(k \frac{\partial \lambda}{\partial k} \right)_u = k \, u_0 \, \varepsilon \, k^{\varepsilon - 1} = \varepsilon \lambda$$

and

$$\beta(u) = \frac{-k(\partial \lambda / \partial k)_u}{(\partial \lambda / \partial u)_k} = \frac{-\varepsilon \lambda}{\frac{\partial}{\partial u}\left(e^{\ln u_0} k^{\varepsilon} \right)_k}$$

$$= \frac{-\varepsilon \lambda}{\left(\frac{\partial \ln u_0}{\partial u} \right) \lambda}$$

$$= -\varepsilon \left(\frac{\partial \ln u_0}{\partial u} \right)^{-1} \qquad (9)$$

$\longrightarrow$ power series in u with ε-dependent coefficients

Similarly, we get

$$\gamma_\phi(u) = \left(k \frac{\partial u}{\partial k} \right) \frac{\partial \ln Z_\phi}{\partial u} = \beta(u) \frac{\partial \ln Z_\phi}{\partial u} \qquad (10)$$

Since (9) contains explicit ε, $\left(\frac{\partial \ln u_0}{\partial u} \right)^{-1}$ must have <u>at most</u> simple poles in ε!

§4.2 Regularization by continuation in the number of dimensions

In a QFT with critical dimension d_0, every term in perturbation series converges when $\Lambda \to \infty$ for $d < d_0$

$\to$ it will converge in a circle with $|d| < d_0$ in complex plane

$\to$ defines analytic continuation as meromorphic function on $\mathbb{C}$

$\to$ poles at set of rational values of d !

Example:

$$I(k) = \int \frac{1}{(q^2+m^2)[(k-q)^2+m^2]}$$

(Exercise)

$$= \left[\frac{1}{2}\Gamma\left(\tfrac{1}{2}d\right)\Gamma\left(2-\tfrac{1}{2}d\right)\right]\left[\frac{\Gamma^2\left(\frac{1}{2}d-1\right)}{\Gamma(d-2)}(k^2)^{(d-4)/2}\right]$$

Exact equality holds for $2<d<4$, RHS has pole for $d=4$. To see this, use

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \frac{1}{12}(6\gamma^2+\pi^2)\varepsilon + \mathcal{O}(\varepsilon^2)$$

Euler's constant ≈ 0.577

$\rightarrow \Gamma(2-\frac{1}{2}d)$ has pole at $d=4$!

Role of renormalization is to cancel these poles.

Massless theory below $d=4$

Consider massless theory:

$$\Gamma_R^{(2)}(k=0) = 0 \qquad (*)$$

(*) does <u>not</u> guarantee that

$$\Gamma_R^{(2)}(k) \xrightarrow[k\to 0]{} 0 \qquad (**)$$

Dimensional analysis gives

$$\Gamma_R^{(2)}(k, g, \kappa) = k^2 F(k, g, \kappa)$$

with $\quad F = \sum_n g^n F_n$

and $\quad F_n = k^{-\varepsilon n} + \underbrace{\ldots}_{\sim k^{-\varepsilon n + i}\kappa^{-i} \text{ (less singular)}}$

$\rightarrow$ if $\varepsilon n > 2$, then $k^2 g^n F_n$ can become divergent for $k \to 0$

$\rightarrow$ in $d<4$ (**) is a property of the sum and <u>not</u> of the individual orders in pert. expansion!

At $d=4$, we have instead:

$$F_n \sim [\ln(k/k)]^n \text{ at most}$$

$\longrightarrow$ $\lim_{k\to 0} k^2 F_n = 0$ always

To approach $d=4$, expand

$$x^\varepsilon = e^{\varepsilon \ln x} = \sum_{n=0}^{\infty} \frac{1}{n!} \varepsilon^n (\ln x)^n$$

$\longrightarrow$ every term in $g^n \varepsilon^n$ has at most logarithmic divergence

$\longrightarrow$ have a well-defined mass-less limit at every order in g